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THERMOELASTIC STRESSES IN A FILAMENT-WOUND ORTHOTROPIC COMPOSITE ELLIPTIC CYLINDER DUE TO A UNIFORM TEMPERATURE CHANGE

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Abstract—The problem of obtaining the thermal stresses and displacements in an orthotropic elliptic cylinder (as can be produced by filament winding on a mandrel of elliptical cross-section) due to a uniform change in temperature, is formulated. The material properties are assumed to be independent of temperature and the anisotropy is referred to the coordinate system that is inherent to the geometry of the filament wound body. A displacement approach is used. Illustrative results are presented for the distribution of stresses and displacements in these orthotropic cylindrical tubes of elliptic cross-section and a comparison with the limit of circular hollow cylinders is made.

INTRODUCTION

An understanding of thermally-induced stresses in anisotropic bodies is essential for a comprehensive study of their response due to an exposure to a temperature change, which may in turn occur in service or during the manufacturing stages. The subject is complicated because of the combined effects of anisotropy and geometry. For example, it is well known that for bodies of rectilinear anisotropy, a temperature field which is linear with respect to the rectangular Cartesian coordinate system results in zero stresses if the body is free of surface tractions and body forces. This result is valid for both simply and multiply connected regions (Boley and Weiner, 1960). Nowacki (1962) noted that for simply connected bodies having cylindrical orthotropy only a uniform temperature distribution results in no stresses, while for simply connected bodies of spherical orthotropy any nonzero temperature variation induces stresses. Hsu and Tauchert (1976) proved that a uniform temperature rise produces no stresses in a cylindrically anisotropic, multiply connected body provided the symmetry axis lies outside the body, but causes stresses when the axis intersects or is surrounded by the body. The latter case is that of circular composite cylinders, and formulations and solutions for the thermal stresses have been presented, for example, by Kalam and Tauchert (1978) and Hyer and Cooper (1986).

Consider now a body produced by filament winding on a mandrel of elliptical cross-section (Fig. 1). Such filament-wound bars could be used in mechanical systems, for example in automotive components. During the curing stages, thermal stresses may be induced from the heat build-up and cooling process. The level of these stresses may well exceed the ultimate strength point. It is the purpose of this paper to present a two-dimensional solution to this problem.

For this geometry, the equivalent directions as regards the elastic properties are the $(\hat{\xi}, \hat{\eta}, \hat{z})$ (Fig. 1), where $\hat{\xi}$ is the direction normal to the layers (thicknesswise direction), $\hat{\eta}$ is the tangent to the periphery and $\hat{z}$ is along the axis of the elliptic cylinder. Hence, this body possesses curvilinear anisotropy referred to the $(\hat{\xi}, \hat{\eta}, \hat{z})$ system. For this system of curvilinear coordinates, the infinitesimal elements isolated by the three pairs of coordinate planes have the same elastic properties. A formulation dealing with the theory of elasticity of elliptic cylinders, produced by filament winding on an elliptical mandrel, was developed by Kardomateas (1988). The type of curvillinear anisotropy inherent in the geometry of the filament-wound body, as described above, was named "elliptical anisotropy". Moreover, the solution was presented for the special case of torsion of an orthotropic body. It should be noted that since a body that is produced by filament winding on the elliptical mandrel

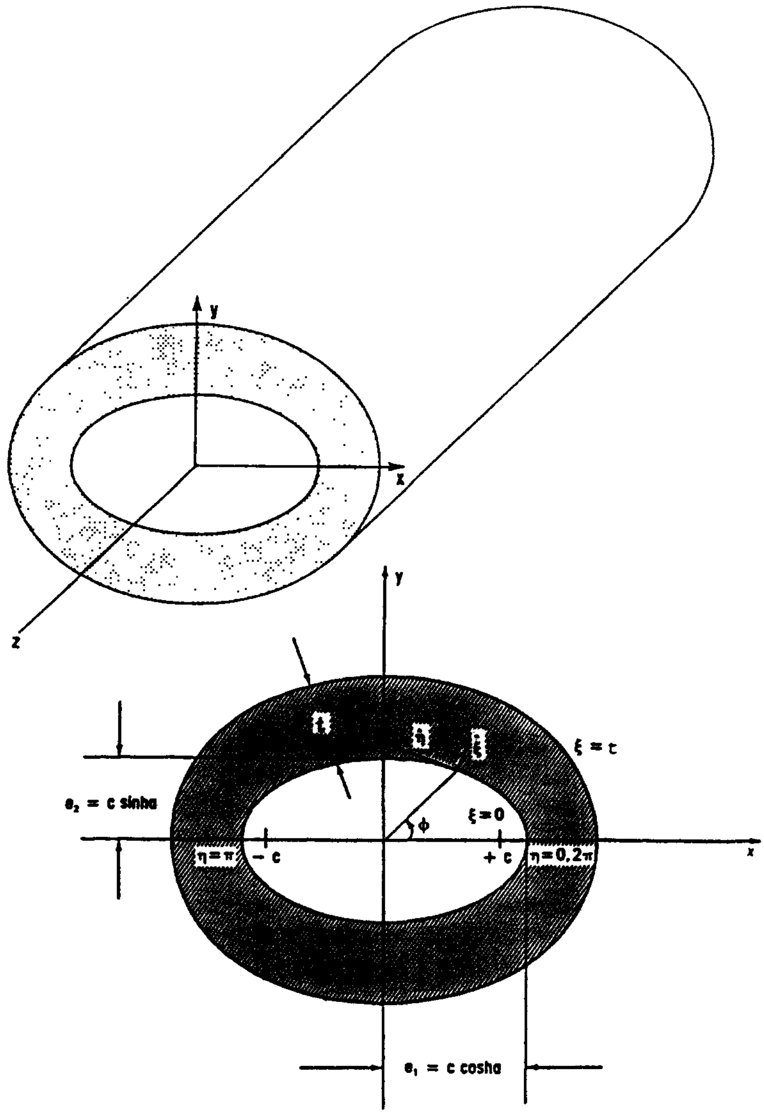

Fig. 1. Definition of the geometry for the filament-wound elliptic cylinder.

has consant thickness, the outer bounding surface is not a true ellipse, as opposed to the internal one ; in the limit of infinite thickness the outer contour is a circle.

In this work the problem of determining the thermally-induced stresses due to a uniform temperature change in a filament-wound orthotropic elliptic cylinder is formulated. It is assumed that the stresses act on the planes normal to the ellipsoid axis and do not vary along the generator and that there are no body forces. After deriving the governing equations for the displacements, numerical results are presented for the variation of the stresses and displacements through the thickness and along the periphery.

FORMULATION

Consider a mandrel of elliptical cross-section with semiaxes $c \cosh a$ and $c \sinh a$ (foci at $\pm c$) (Fig. 1). For the body that is produced by filament winding on that mandrel, the coordinate directions that coincide with the directions which are equivalent in the sense of the elastic properties are the $(\hat{\xi}, \hat{\eta}, \hat{z})$. We should therefore refer to this system of curvilinear coordinates. On a Cartesian coordinate system, the filament-would body is defined by taking the definition for the x and y coordinates of the elliptical mandrel and the normal unit vector, which gives (Fig. 1),

$$x = c\cosh a\cos\eta + \xi\,\frac{c}{h}\sinh a\cos\eta \tag{1a}$$

$$y = c\sinh a\sin\eta + \xi\,\frac{c}{h}\cosh a\sin\eta \tag{1b}$$

where

$$h = h(\eta) = c\sqrt{\sinh^2 a + \sin^2\eta}. \tag{2}$$

Different values of ξ correspond to different layers on the elliptical mandrel. So ξ varies from 0 to t, where t is the total thickness of the body. On any of these layers ξ is constant and η varies through a range of 2π.

It is assumed that the whole body undergoes a constant temperature change ΔT. An orthotropic material would physically characterize the effective behavior of a bar made of a unidirectional fiber-reinforced material where the fibers are oriented at a constant 90° to the z-axis. The constitutive equations with respect to the curvilinear coordinate system $(\hat{\xi}, \hat{\eta}, \hat{z})$ are :

$$\begin{bmatrix} \sigma_{\xi\xi} \\ \sigma_{\eta\eta} \\ \sigma_{zz} \\ \tau_{\eta z} \\ \tau_{\xi z} \\ \tau_{\xi\eta} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{\xi\xi} - \alpha_\xi \Delta T \\ \varepsilon_{\eta\eta} - \alpha_\eta \Delta T \\ \varepsilon_{zz} - \alpha_z \Delta T \\ \gamma_{\eta z} \\ \gamma_{\xi z} \\ \gamma_{\xi\eta} \end{bmatrix} \tag{3}$$

where C_{ij} are the elastic constants and α_i the thermal expansion coefficients.

Since the temperature does not depend on the axial coordinate, we can assume that the stresses are independent of z. Therefore, we are focusing on the plane problem. In addition to the constitutive eqns (1), the elastic response of the elliptic cylinder must satisfy the equilibrium equations (Kardomateas, 1988):

$$q\sigma_{\xi\xi,\xi} + \tau_{\xi\eta,\eta} + \frac{\partial q}{\partial \xi}(\sigma_{\xi\xi} - \sigma_{\eta\eta}) = 0 \tag{4a}$$

$$\sigma_{\eta\eta,\eta} + q\tau_{\xi\eta,\xi} + 2\frac{\partial q}{\partial \xi}\tau_{\xi\eta} = 0 \tag{4b}$$

$$(q\tau_{\xi z})_{,\xi} + \tau_{\eta z,\eta} = 0 \tag{4c}$$

where

$$q = \left[\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2\right]^{1/2} = h + \xi\,\frac{c^2 \sinh 2a}{2h^2}. \tag{5}$$

The strain–displacement relations are :

$$\varepsilon_{\xi\xi} = u_{\xi,\xi}\ ;\quad \varepsilon_{\eta\eta} = \frac{1}{q}u_{\eta,\eta} + \frac{1}{q}\frac{\partial q}{\partial \xi}u_\xi \tag{6a}$$

$$\varepsilon_{zz} = u_{z,z}\ ;\quad \gamma_{\eta z} = u_{\eta,z} + \frac{1}{q}u_{z,\eta}\ ;\quad \gamma_{\xi z} = u_{\xi,z} + u_{z,\xi} \tag{6b}$$

$$\gamma_{\xi\eta} = \frac{1}{q}u_{\xi,\eta} + u_{\eta,\xi} - \frac{1}{q}\frac{\partial q}{\partial \xi}u_\eta. \tag{6c}$$

For the problem without the thermal effects, the expressions for the displacement field were derived by Kardomateas (1988). A similar procedure can be followed in this thermoelastic problem. In particular, define by a_{ij} the elastic compliance constants, i.e. the components of the inverse matrix of $[C]$ in (3), so that the strains are expressed in terms of stresses. Now set :

$$D(\xi,\eta) = a_{13}\sigma_{\xi\xi}+a_{23}\sigma_{\eta\eta}+a_{33}\sigma_{zz}+\alpha_z\Delta T. \tag{7}$$

Integrate the strain–displacement eqns (6b), substitute the resulting expressions and the expressions for the strains in terms of the stress components in the remaining strain–displacement eqns (6a), (6c) and equate equal powers of z. This yields differential equations for $D(\xi,\eta)$ and the displacement functions in the same manner as in Kardomateas (1988), which finally yield for D :

$$D(\xi,\eta) = \frac{A}{c}x(\xi,\eta)+\frac{B}{c}y(\xi,\eta)+C \tag{8}$$

and lead to the general solution for the displacements in this thermoelastic problem as follows :

$$\begin{aligned} u_\xi = U(\xi,\eta)-\frac{z^2}{2}\frac{1}{h}(A\sinh a\cos\eta+B\cosh a\sin\eta) \\ +z\bar{\theta}\frac{\partial h}{\partial\eta}-z\frac{c}{h}(\omega_x\cosh a\sin\eta-\omega_y\sinh a\cos\eta) \\ +\frac{c}{h}(v_{0x}\sinh a\cos\eta+v_{0y}\cosh a\sin\eta)+\frac{\partial h}{\partial\eta}\omega_z \end{aligned} \tag{9a}$$

$$\begin{aligned} u_\eta = V(\xi,\eta)+\frac{z^2}{2}\frac{1}{h}(A\cosh a\sin\eta-B\sinh a\cos\eta) \\ +z\bar{\theta}\left(\xi+h\frac{\partial q}{\partial\xi}\right)-z\frac{c}{h}(\omega_x\sinh a\cos\eta+\omega_y\cosh a\sin\eta) \\ +\frac{c}{h}(-v_{0x}\cosh a\sin\eta+v_{0y}\sinh a\cos\eta)+\left(\xi+h\frac{\partial q}{\partial\xi}\right)\omega_z \end{aligned} \tag{9b}$$

$$\begin{aligned} u_z = W(\xi,\eta)+z\left[A\left(\cosh a+\frac{\xi}{h}\sinh a\right)\cos\eta+B\left(\sinh a+\frac{\xi}{h}\cosh a\right)\sin\eta+C\right] \\ +\xi\frac{c}{h}(\omega_x\cosh a\sin\eta-\omega_y\sinh a\cos\eta) \\ +c(\omega_x\sinh a\sin\eta-\omega_y\cosh a\cos\eta)+v_{0z}. \end{aligned} \tag{9c}$$

Notice that the general expressions for the components of the displacement field are essentially unaffected by the thermal expansion terms. In the above expressions, the functions U, V, W represent the displacements accompanied by deformation and the constants v_{0x}, v_{0y}, v_{0z}, ω_x, ω_y, ω_z characterize the rigid body translation and rotation about the Cartesian coordinate system. The constants A, B, C, $\bar{\theta}$ depend on the boundary conditions, as discussed later. The strains for this displacement field are given by

$$\varepsilon_{\xi\xi} = U_{,\xi}\ ;\quad \varepsilon_{\eta\eta} = \frac{1}{q}\left(V_{,\eta} + \frac{\partial q}{\partial \xi} U\right);\quad \gamma_{\xi\eta} = \frac{1}{q}\left(U_{,\eta} - \frac{\partial q}{\partial \xi} V + qV_{,\xi}\right) \tag{10a}$$

$$\varepsilon_{zz} = A\left(\cosh a + \frac{\xi}{h}\sinh a\right)\cos\eta + B\left(\sinh a + \frac{\xi}{h}\cosh a\right)\sin\eta + C \tag{10b}$$

$$\gamma_{\eta z} = \frac{1}{q} W_{,\eta} + \bar{\theta}\left(\xi + h\frac{\partial q}{\partial \xi}\right);\quad \gamma_{\xi z} = W_{,\xi} + \bar{\theta}\frac{\partial h}{\partial \eta}. \tag{10c}$$

Using a displacement approach, substitute (3) and (10) into the equilibrium eqns (4) to obtain the governing field equations for U, V, W:

$$C_{11}\left(qU_{,\xi\xi} + \frac{\partial q}{\partial \xi}U_{,\xi}\right) + \frac{C_{66}}{q}\left(U_{,\eta\eta} - \frac{1}{q}\frac{\partial q}{\partial \eta}U_{,\eta}\right) - \frac{C_{22}}{q}\left(\frac{\partial q}{\partial \xi}\right)^2 U$$

$$+ (C_{12} + C_{66})V_{,\xi\eta} - \left(\frac{C_{22} + C_{66}}{q}\right)\frac{\partial q}{\partial \xi}V_{,\eta} + \frac{3C_{66}}{q^2}\frac{\partial q}{\partial \xi}\frac{\partial h}{\partial \eta}V = P_1(\xi,\eta) \tag{11a}$$

$$(C_{12} + C_{66})U_{,\xi\eta} + \frac{1}{q}\frac{\partial q}{\partial \xi}(C_{22} + C_{66})U_{,\eta} - \frac{3C_{22}}{q^2}\frac{\partial q}{\partial \xi}\frac{\partial h}{\partial \eta}U$$

$$+ C_{66}\left[qV_{,\xi\xi} + \frac{\partial q}{\partial \xi}V_{,\xi} - \frac{1}{q}\left(\frac{\partial q}{\partial \xi}\right)^2 V\right] + \frac{C_{22}}{q}\left(V_{,\eta\eta} - \frac{1}{q}\frac{\partial q}{\partial \eta}V_{,\eta}\right) = P_2(\xi,\eta) \tag{11b}$$

$$C_{55}\left(qW_{,\xi\xi} + \frac{\partial q}{\partial \xi}W_{,\xi}\right) + \frac{C_{44}}{q}\left(W_{,\eta\eta} - \frac{1}{q}\frac{\partial q}{\partial \eta}W_{,\eta}\right) = P_3(\xi,\eta) \tag{11c}$$

where

$$P_1(\xi,\eta) = \frac{\partial q}{\partial \xi}[(C_{11} - C_{12})\alpha_\xi + (C_{12} - C_{22})\alpha_\eta + (C_{13} - C_{23})\alpha_z]\Delta T$$

$$+ \frac{\partial q}{\partial \xi}(C_{23} - C_{13})\left[A\left(\cosh a + \frac{\xi}{h}\sinh a\right)\cos\eta + B\left(\sinh a + \frac{\xi}{h}\cosh a\right)\sin\eta + C\right]$$

$$- C_{13}\frac{q}{h}(A\sinh a\cos\eta + B\cosh a\sin\eta) \tag{12a}$$

$$P_2(\xi,\eta) = C_{23}\frac{q}{h}(A\cosh a\sin\eta - B\sinh a\cos\eta) \tag{12b}$$

$$P_3(\xi,\eta) = (C_{44} - C_{55})\bar{\theta}\frac{\partial q}{\partial \xi}\frac{\partial h}{\partial \eta}. \tag{12c}$$

Notice that the equation governing the axial displacement component, W, is uncoupled from the other two equations (this is not true in the generally anisotropic case). Hence, we can set $\bar{\theta} = W = 0$ and restrict our attention to the two eqns (11a, b) for U, V. The components of displacements inside the body must be continuous and single-valued functions of the coordinates. For their determination, the above second-order differential equations are solved subject to the boundary conditions which will be considered next.

We assume that no external tractions exist. Then the conditions on the contour bounding the cross-section (outmost and innermost layers) can be written in the following form: $\sigma_{\xi\xi} = \tau_{\xi\eta} = 0$. These conditions are written as follows in terms of the displacements.

For $\xi = 0$ and $\xi = t$,

$$C_{11}U_{,\xi}+\frac{C_{12}}{q}\left(V_{,\eta}+\frac{\partial q}{\partial \xi}U\right)-(C_{11}\alpha_\xi+C_{12}\alpha_\eta)\Delta T$$
$$+C_{13}\left[A\left(\cosh a+\frac{\xi}{h}\sinh a\right)\cos\eta\right.$$
$$\left.+B\left(\sinh a+\frac{\xi}{h}\cosh a\right)\sin\eta+C\right]-C_{13}\alpha_z\Delta T=0 \quad (13a)$$

and

$$\frac{C_{66}}{q}\left(U_{,\eta}-\frac{\partial q}{\partial \xi}V+qV_{,\xi}\right)=0. \quad (13b)$$

Now, let us consider the conditions of resultant forces and moments. Since the stresses do not depend on z, these conditions exist in any cross-section. It was proved (Kardomateas, 1988) that the conditions of zero resultant forces along the x- and y-axes are satisfied identically. Therefore the following conditions remain :

$$\iint \sigma_{zz}\,\mathrm{d}S=P_z=0\,;\quad \iint \sigma_{zz}y\,\mathrm{d}S=M_1=0\,;\quad \iint \sigma_{zz}x\,\mathrm{d}S=M_2=0 \quad (14a)$$

$$\iint\left[\left(\xi+h\frac{\partial q}{\partial \xi}\right)\tau_{\eta z}+\frac{\partial h}{\partial \eta}\tau_{\xi z}\right]\mathrm{d}S=M_t=0. \quad (14b)$$

In the above expressions, P_z is the resultant axial force, M_1, M_2 are the bending moments about the x-, y-axes, respectively, and M_t is the twisting moment.

Therefore, by substituting (3) and (10) in (14), we find that for a body with a bounded cross-section on the ends (and in any cross-section) the following conditions must be satisfied :

$$(C_{23}-C_{13})\int_{\xi=0}^{T}\int_{\eta=0}^{2\pi}\frac{\partial q}{\partial \xi}U\,\mathrm{d}\xi\,\mathrm{d}\eta+C_{13}\int_{\eta=0}^{2\pi}\left[\left(h+T\frac{c^2\sinh 2a}{2h^2}\right)U(T,\eta)-hU(0,\eta)\right]\mathrm{d}\eta$$
$$-(C_{13}\alpha_\xi+C_{23}\alpha_\eta+C_{33}\alpha_z)S(\Delta T)+C_{33}CS=0 \quad (15a)$$

$$(C_{23}-C_{13})\int_{\xi=0}^{T}\int_{\eta=0}^{2\pi}y\frac{\partial q}{\partial \xi}U\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+C_{13}\int_{\eta=0}^{2\pi}\left[U(t,\eta)\left(h+T\frac{c^2\sinh 2a}{2h^2}\right)c\sin\eta\left(\sinh a+\frac{t}{h}\cosh a\right)\right.$$
$$\left.-U(0,\eta)hc\sinh a\sin\eta\right]\mathrm{d}\eta-\int_{\xi=0}^{t}\int_{\eta=0}^{2\pi}\frac{qc}{h}(C_{13}U\cosh a\sin\eta$$
$$+C_{23}V\sinh a\cos\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta+C_{33}BI_1/c=0 \quad (15b)$$

$$(C_{23}-C_{13})\int_{\xi=0}^{T}\int_{\eta=0}^{2\pi}x\frac{\partial q}{\partial \xi}U\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+C_{13}\int_{\eta=0}^{2\pi}\left[U(t,\eta)\left(h+T\frac{c^2\sinh 2a}{2h^2}\right)c\cos\eta\left(\cosh a+\frac{t}{h}\sinh a\right)\right.$$
$$\left.-U(0,\eta)hc\cosh a\cos\eta\right]\mathrm{d}\eta-\int_{\xi=0}^{t}\int_{\eta=0}^{2\pi}\frac{qc}{h}(C_{13}U\sinh a\cos\eta$$
$$-C_{23}V\cosh a\sin\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta+C_{33}AI_2/c=0. \quad (15c)$$

The above expressions are in terms of the cross-sectional area S and the principal moments of inertia with respect to the x- and y-axes, I_1, I_2, which are assumed to pass through the centroid of the section (Fig. 1),

$$S = \int_{\xi=0}^{T} \int_{\eta=0}^{2\pi} q \, d\xi \, d\eta \tag{16a}$$

$$I_1 = \int_{\xi=0}^{T} \int_{\eta=0}^{2\pi} y^2 q \, d\xi \, d\eta \; ; \quad I_2 = \int_{\xi=0}^{T} \int_{\eta=0}^{2\pi} x^2 q \, d\xi \, d\eta. \tag{16b}$$

An important observation can be made by considering the structure of eqns (11). Specifically, the symmetry reduces the problem to only a quarter ellipse. For the other points around the periphery, the displacements should have the following symmetry: For $\pi/2 \leqslant \eta \leqslant \pi$, $U(\xi,\eta) = U(\xi,\pi-\eta)$; $V(\xi,\eta) = -V(\xi,\pi-\eta)$. For $\pi \leqslant \eta \leqslant 3\pi/2$, $U(\xi,\eta) = U(\xi,\eta-\pi)$; $V(\xi,\eta) = V(\xi,\eta-\pi)$, and for $3\pi/2 \leqslant \eta \leqslant 2\pi$, $U(\xi,\eta) = U(\xi,2\pi-\eta)$; $V(\xi,\eta) = -V(\xi,2\pi-\eta)$. Taking this into account we conclude from (15b, c) that the constants $A = B = 0$, and so under these circumstances *only the unknown C remains* and out of the three conditions (15) only eqn (15a) needs to be satisfied.

Thus the problem is reduced to that for a quarter ellipse. The unknowns are the values of U, V at the interior and boundary points as well as the value of the constant C. The equations involved are the two differential eqns (11a, b), the boundary conditions (13a, b) and the resultant force condition (15a). Due to the highly nonlinear nature of the equations, an analytical solution could not be found. For a numerical solution, the finite-difference method is used. Despite the reduction due to the symmetry considerations, the number of equations is still large enough. Considerable reduction in the required computer capacity is achieved if the coefficient matrix is banded. However, a regular banded coefficient matrix occurs only if the constant C is assumed known. To this extent, a special sparse system solving technique was developed by using a Gauss elimination scheme for the solution of the resulting nonsymmetric banded system of linear equations in which the coefficient matrix has in addition a nonzero end row and column. In this manner the displacements as well as the unknown constant C can be found directly. If we assume that we have M_q points around the periphery of the quarter ellipse then the semibandwidth of the system is $4(M_q+1)$.

DISCUSSION OF RESULTS

The distribution of thermal stresses was determined for a body produced by filament winding on a mandrel of elliptical cross-section with major and minor semiaxes $e_1 =$ 40 mm and $e_2 = 20$ mm (Fig. 1). The elliptic cylinder produced in this manner is assumed to have a thickness of $t = 15$ mm. The material is graphite/expoxy and it is laid so that the fibers are directed along the periphery (η direction). The stiffness coefficients are taken as follows in MN m^{-2} (where $1 \equiv \xi$, $2 \equiv \eta$, $3 \equiv z$) : $C_{11} = 13{,}700$, $C_{22} = 150{,}500$, $C_{33} = 12{,}550$, $C_{23} = 5787$, $C_{13} = 6828$, $C_{12} = 6105$, $C_{66} = 4690$. The thermal expansion coefficients are taken to be $a_\xi = a_z = 33.7 \times 10^{-6}$ per °C, $a_\eta = -0.077 \times 10^{-6}$ per °C. A uniform temperature increase of 100°C was used for loading.

Figure 2 shows the distribution of $\sigma_{\xi\xi}$ through the thickness at the polar angle locations $\phi = 0°$, 45° and 90°. This stress is compressive of increasing magnitude towards the major axis. Figure 3 shows the thicknesswise distribution of $\sigma_{\eta\eta}$ at the angular sites $\phi = 0°$ and 90°. This component of stress (along the direction of fibers), which is of the biggest magnitude, is compressive below a certain point through the thickness and tensile thereafter. It has a maximum compressive value at the inside surface $\xi = 0$ and a maximum tensile value at the outside surface $\xi = t$. Depending on the location through the thickness this stress may be decreasing or increasing in magnitude as we proceed along the periphery from the major to the minor axis. The distribution of the axial stress σ_{zz} versus ξ is illustrated in Fig. 4 for the angular locations $\phi = 0°$, 45° and 90°. This component of stress is compressive

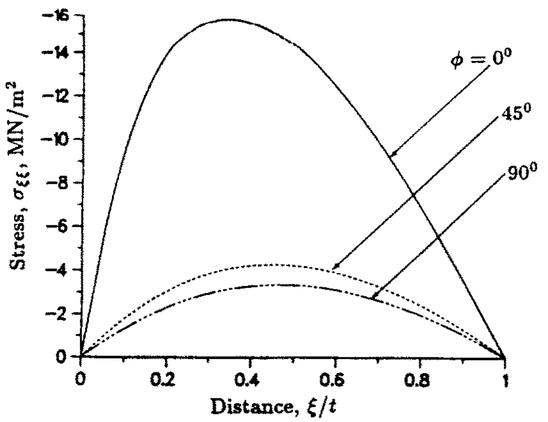

Fig. 2. Through the thickness distribution of the stress $\sigma_{\xi\xi}$ at the angular locations $\phi = 0^\circ$ (major axis), 45°, and 90° (minor axis).

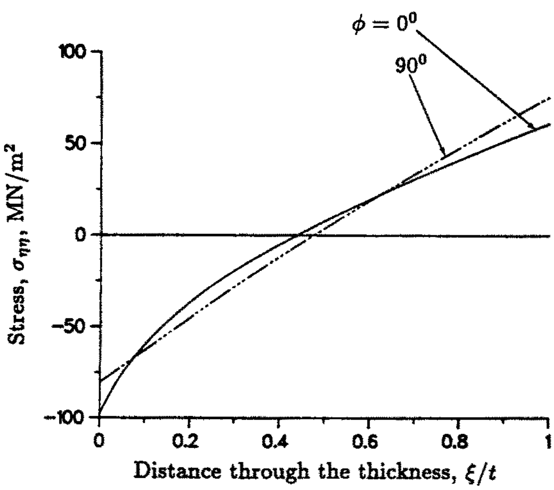

Fig. 3. Through the thickness (along ξ) distribution of the stress $\sigma_{\eta\eta}$ at the angular locations $\phi = 0^\circ$ (major axis) and 90° (minor axis).

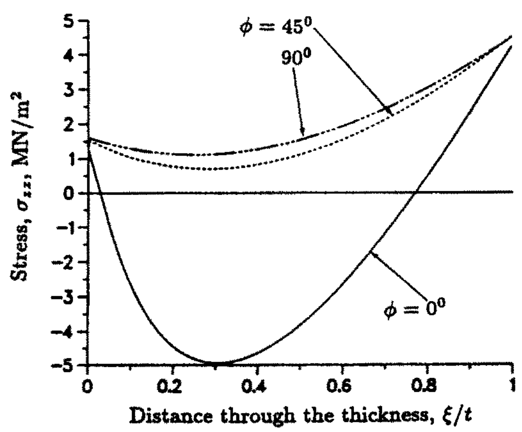

Fig. 4. Thicknesswise distribution of the stress σ_{zz} at the angular locations $\phi = 0^\circ$ (major axis), 45°, and 90° (minor axis).

at the major axis, $\phi = 0^\circ$, except near the inside and outside boundaries ; it becomes tensile as we proceed towards the minor ($\phi = 90^\circ$) axis. Figure 5 shows the variation of the shear stress $\tau_{\xi\eta}$ with the polar angle ϕ at the locations through the thickness $\xi/T = 0.25$, 0.5 and 0.75. This stress has a tensile peak much larger than the corresponding compressive one

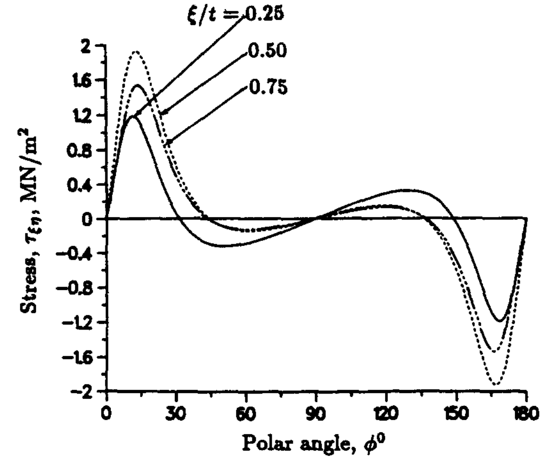

Fig. 5. Angular distribution of the shear stress $\tau_{\xi\eta}$ at the locations through the thickness $\xi/t = 0.25$, 0.50 and 0.75. Notice that $\tau_{\xi\eta}$ would be zero throughout for a circular cross-section.

and is zero at the major and minor axes due to symmetry. Notice that this component of stress would not exist for a circular cylindrical tube.

Figure 6 illustrates the thicknesswise variation of the displacement component U at $\phi = 0°, 45°$ and $90°$. The displacement is varying in a more linear fashion as the minor axis is approached and is increasing in algebraic value from the major to the minor axis and from the inside to the outside boundary. The distribution of the other displacement component V along the periphery is shown in Fig. 7 for locations through the thickness $\xi/T = 0, 0.5, 1.0$. This displacement component is zero at the major and minor axis due to symmetry. Notice that V would not exist if the cross-section were a circle.

Now let us consider the limit of the elliptical mandrel becoming a circular cylinder of radius r_1. The resulting filament-wound body will be a circular cylindrical tube of radii r_1 and r_2 and will possess cylindrical orthotropy (referred to a polar coordinate system r, θ, z). In this case $\xi \equiv r$, $\eta \equiv \theta$, $z \equiv z$, $q \equiv r$. Furthermore, $\gamma_{r\theta} = 0$ and only the displacement component $U(r)$ exists and the solution is given as follows (Avery and Herakovich, 1986, or Kardomateas, 1989): for $C_{11} \neq C_{22}$,

$$U(r) = G_1 r^{\lambda_1} + G_2 r^{\lambda_2} + \frac{C_{23} - C_{13}}{C_{11} - C_{22}} Cr + \frac{b(\Delta T)}{C_{11} - C_{22}} r \tag{17a}$$

where $\lambda_{1,2} = \pm\sqrt{C_{22}/C_{11}}$ and

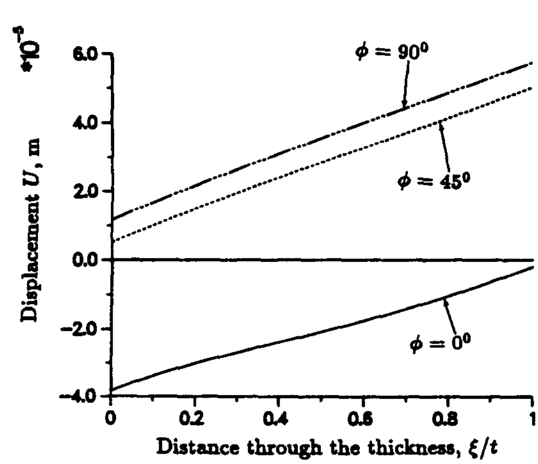

Fig. 6. Through the thickness distribution of the displacement U at the angular locations $\phi = 0°$ (major axis), 45°, and 90° (minor axis).

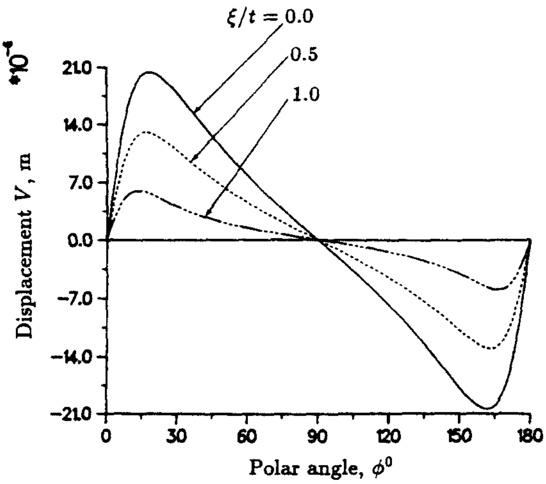

Fig. 7. Angular distribution of the displacement V at the locations through the thickness $\xi/t = 0.0$ (inside boundary), 0.5, and 1.0 (outside boundary). Notice that this displacement component would be zero throughout for a circular cross-section.

$$b = (C_{11} - C_{12})\alpha_r + (C_{12} - C_{22})\alpha_\theta + (C_{13} - C_{23})\alpha_z. \tag{17b}$$

For $C_{11} = C_{22}$ the corresponding solution for the displacement is

$$U(r) = G_1 r + \frac{G_2}{r} + \frac{(C_{23} - C_{13})}{2C_{11}} Cr \ln(r/r_2) + \frac{b(\Delta T)}{2C_{11}} r \ln(r/r_2). \tag{17c}$$

Notice that in the above expressions the constant C has the same meaning as in the present case [eqn (10b)], i.e. it is the constant value of the axial strain ε_{zz}. The constants G_1, G_2, C are found from the two conditions of zero traction at the bounding surfaces, $\sigma_{rr}(r_{1,2}) = 0$, and the condition of zero resultant axial force $\int_{r_1}^{r_2} \sigma_{zz}(r) 2\pi r \, dr = 0$ (Avery and Herakovich, 1986, or Kardomateas, 1989).

To compare with the solution for a cylindrical tube, the major axis of the mandrel was kept at 40 mm while the minor axis was increased until it became equal to the major one (the limit of a circular section). Figure 8 shows the effect of the variation of the major to minor axis ratio on the stress $\sigma_{\xi\xi}$ at $\xi/t = 0.5$. As the elliptical cross-section of the mandrel

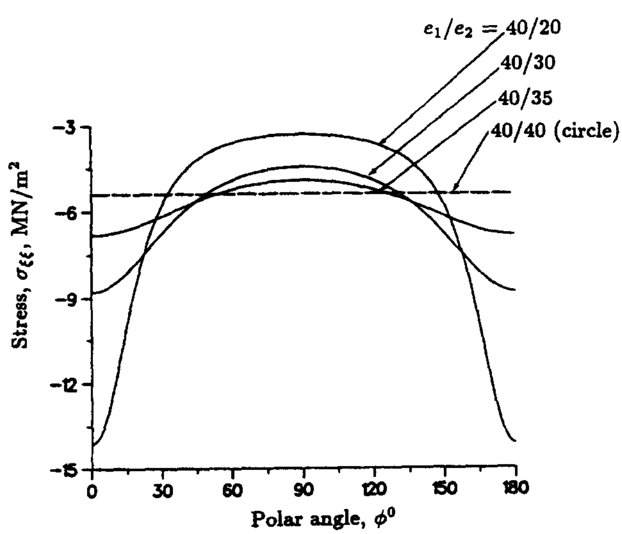

Fig. 8. The effect of the decrease in the major to minor axis ratio (e_1/e_2) of the mandrel on the angular distribution of the stress $\sigma_{\xi\xi}$ at the location through the thickness $\xi/t = 0.5$. The broken line is the solution for a circular cross-section.

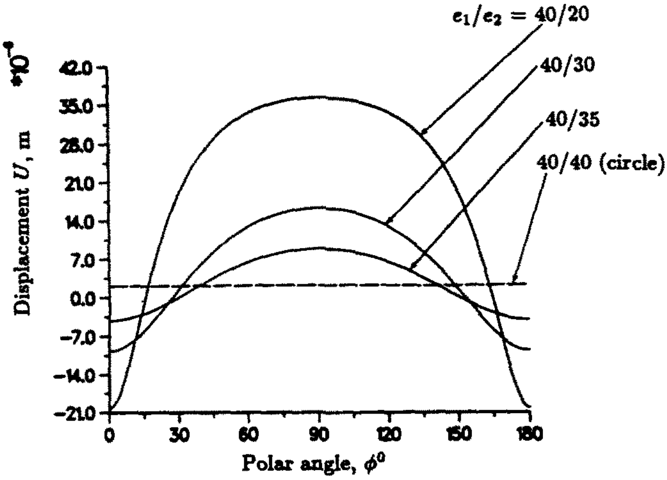

Fig. 9. The effect of the decrease towards unity of the major to minor axis ratio (e_1/e_2) of the mandrel on the angular distribution of the displacement U (location through the thickness, $\xi/t = 0.5$). The broken line is the solution for a circular cross-section.

approaches a circle, the distribution along the periphery becomes flatter. It is seen that the solution converges to the constant value (for a circular cross-section there is no angular dependence) that would be given by the solution described above for a circular cylindrical tube (dashed line). In a similar fashion, Fig. 9 shows the angular variation of U at $\xi/t = 0.5$ for decreasing ratio of major to minor semiaxis, e_1/e_2, until the limit of unity (a circle, dashed line). The distribution similarly becomes flatter converging to the constant value for the limit of a circular section.

CONCLUSIONS

An analysis has been presented for obtaining the stresses and deformations of filament-wound elliptic cylinders subjected to a uniform temperature change. The body is assumed to consist of orthotropic material with properties independent of temperature. The anisotropy in this case is curvilinear, defined with respect to the directions that are equivalent in the sense of the elastic properties. Numerical results for the distribution of stresses and displacements through the thickness and along the periphery were presented and discussed. These distributions for the elliptic cylinder converge to the corresponding ones for the limit of a circular hollow cylinder, as the major over minor axis ratio tends to unity.

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